

Properties of the Schwinger series and pair creation in strong fields

B. B. Levchenko*

*Skobeltsyn Institute of Nuclear Physics,
M.V. Lomonosov Moscow State University,
119991 Moscow, Russian Federation*

Probabilities of a pair of fermions and bosons creation in a static and spatially uniform electric field E are represented in the Schwinger formulas by infinite series. It is believed that in weak fields the main contribution to the probability is given by the first term of series, however the size of the remainder apparently was analyzed by nobody. We study the mathematical structure of the Schwinger series by using methods developed during last decades and prove that the given series allows an exact summation and the contribution of remainder grows rapidly with the field strength. As a rule, it is argued that the pair of particles begin to be produced efficiently from the vacuum only in the fields of strength $E \geq E_{cr}$. However, the direct calculation shows that the Schwinger formula for creation of e^+e^- pairs is valid only at the field intensities of $E < 0.0291E_{cr}$. At higher fields, the probability of pair production in an unit space-time volume exceeds unity. In this regard, we refine the formula for the probability of pair creation and numerically find that in the field of strength 2.95% of E_{cr} the pair production probability is almost 100%.

1. More than sixty years ago Julian Schwinger [1] published the formulas that allow to calculate the effects from impact of strong electromagnetic fields on the vacuum. We studied the vacuum polarization in strong electric \mathbf{E} and magnetic \mathbf{H} fields, and the resulting one-loop non-linear correction to the Maxwell's Lagrange function. Of this fundamental paper on quantum electrodynamics we are interested in a particular result obtained "incidentally" by a modest statement of Schwinger [1]. It was found that the change in the Lagrangian density causes the appearance of imaginary part of the effective action that corresponds to a non-zero probability of actual pair creation in the field. The probability $\bar{w}^{(s)}$, per unit time and per unit volume, that a pair of particles carrying spin s is created by a homogeneous and static electric field is expressed by [1]

$$\bar{w}^{(s)}(E) = (2s+1) \frac{\alpha}{2\pi^2} E^2 \sum_{n=1}^{\infty} \frac{(\pm 1)^{n+1}}{n^2} e^{-\pi n E_{cr}/E}, \quad (1)$$

where¹ $\alpha = e^2/4\pi$, $E_{cr} = m^2/e$ is the so-called critical field, m the particle mass, e the charge on the particle. The '+' sign corresponds to creation of fermion pairs, and the '-' sign for production of bosons. At production of e^+e^- pairs, $E_{cr} = 1.32 \times 10^{16}$ V/cm. The scale factor E_{cr} was introduced and evaluated for the first time in 1931 by Fritz Sauter [2]. Equation (1) corresponds to the weak field approximation.

The peak electric field achieved at the focus of today's most powerful lasers still several orders of magnitude lower E_{cr} . This is the main obstacle for a direct experimental verification of given fundamental mechanism of particle production. Despite of this, the Schwinger mechanism from QED, was the basis for successful phenomenological models of hadron production in collisions

at high energies via the tunnel creation of $q\bar{q}$ -pairs in string-like chromoelectric field [3–5].

From the form in which presented the result (1) and the value of E_{cr} , it seems self-evident that in weak fields the main contribution to the production probability gives the first term of the series

$$\bar{w}^{(1/2)} \approx \frac{\alpha}{\pi^2} E^2 \exp \left\{ -\pi \frac{E_{cr}}{E} \right\}, \quad (2)$$

and this probability is very small. It should be noted, however, that the total value of residual terms of an infinite series, apparently not analyzed. One of the purposes of this article² is an evaluation how strongly change the remainder of the series (1) with increasing field strength. The analysis of this formal mathematical problem allowed to find the exact sum of the infinite series (1), and find a number of unexpected consequences, not mentioned by other authors.

2. To solve the above problem we introduce the reduced field $\beta = E/E_{cr}$, and denote the exponential factor in (1) by $x = \exp(-\pi/\beta)$. As a result, (1) takes the form

$$\bar{w}^{(s)} = \pm (2s+1) \frac{\alpha}{2\pi^2} E_{cr}^2 \beta^2 \sum_{n=1}^{\infty} \frac{(\pm x)^n}{n^2}. \quad (3)$$

Analysis of the mathematical literature has shown that the series in (3) belongs to a class of polylogarithms, and in our particular case we are dealing with the dilogarithm. In mathematics, the dilogarithm studied since Leibniz and Euler, but in the last quarter of the 20th century polylogarithms and hyperlogarithm again attracted attention and appeared in many branches of mathematics and physics. Polylogarithms and reference information of

*Electronic address: levchen@mail.desy.de

¹ Schwinger employs units in which $\hbar = c = 1$.

² Based on a report presented at the conference "Lomonosov Readings", MSU, Moscow, Russia, 14 Nov 2011.

TABLE I: The exact values of the Rogers dilogarithm

x	$-\infty$	-1	0	$\frac{(\sqrt{5}-1)^2}{4}$	$\frac{1}{2}$	$\frac{\sqrt{5}-1}{2}$	1	∞
$L(x)$	$-\frac{\pi^2}{6}$	$-\frac{\pi^2}{12}$	0	$\frac{\pi^2}{15}$	$\frac{\pi^2}{12}$	$\frac{\pi^2}{10}$	$\frac{\pi^2}{6}$	$\frac{\pi^2}{3}$

them are included in the standard set of special functions in most packages of applied mathematical programs.

Let us write out the definition of the dilogarithm and its integral representation following the interesting original review [6],

$$\text{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2} = - \int_0^x \frac{\ln(1-t)}{t} dt. \quad (4)$$

Thus, in terms of the function $\text{Li}_2(x)$ the probability of pair creation we rewrite as follows,

$$\bar{w}^{(s)}(\beta) = \pm(2s+1) \frac{\alpha}{2\pi^2} E_{cr}^2 \beta^2 \text{Li}_2(\pm e^{-\pi/\beta}), \quad (5)$$

and the problem of summing the infinite series in (1) can be regarded as solved. However, to clarify the meaning of this result, let us recall some properties of the dilogarithm [6]. In this case it is more convenient and more compact to use the Rogers dilogarithm $L(x)$, $0 < x < 1$, defined as

$$L(x) = \text{Li}_2(x) + \frac{1}{2} \ln(x) \ln(1-x). \quad (6)$$

The function $L(x)$ allows an analytical continuation into the complex plane with cuts $(-\infty, 0]$ and $[1, +\infty)$ along the real axis. Using a number of theorems and functional relations, $L(x)$ can be extended to the whole real axis. There are only few real points $\{x_i\}$, where the values of $L(x)$ are known exactly. Their values are summarized in Table I. In general, for more than two centuries of research were opened a large number of functional identities, allowing to relate dilogarithms with different powers of x . There are identities and for more complicated rational arguments.

Coming back to the problem of calculating the probability of pair creation by (5), we summarize that by using the integral representation of $\text{Li}_2(x)$, one can calculate $\bar{w}^{(s)}(\beta)$ with any given accuracy, controlled by the precise values of $L(x_i)$ from Table I. Another approach is possible also. Having determined numerically $L(x)$ for a given x_0 , the values of $L(x)$ at other points are restored through a chain of functional identities.

To conclude this section we estimate, at which values of β the remainder of the series (1) starts to give a significant increment to (2). We consider only the case of e^+e^- pairs. To do this, compose a ratio of the total Schwinger

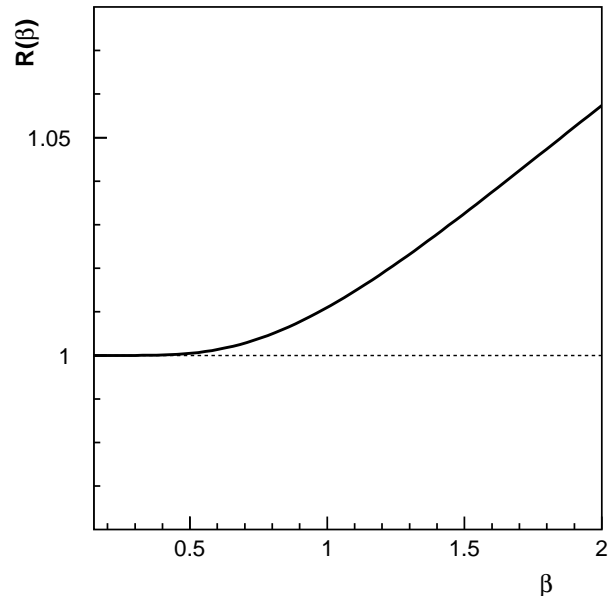


FIG. 1: e^+e^- pair production. The ratio of the total Schwinger series (1) to its first term (2) as a function of the reduced field β .

series to its first term

$$R(\beta) = \frac{\text{Li}_2(e^{-\pi/\beta})}{e^{-\pi/\beta}}. \quad (7)$$

By a numerical integration one find (see Fig. 1) that only at $\beta > 0.5$ the remainder of the series begins to play an important role, making the correction of 6% at $\beta = 2$. But these values of β are beyond the weak field approximation. Thus, (2) indeed is a very good approximation to (1).

3. The method of the simplified record of results when a group of some physical constants are set equal to unity is widespread in theoretical physics. In the case under study (1) it is $\hbar = c = 1$. As noted by several authors [7], "simplifications" achieved in the natural system of units (the system of Planck and Hartree, a relativistic system), sometimes are fraught with illusions³ and Schwinger for-

³ In this regard, in the book [7] on page 97 there is a remarkable comment: "Thus, simplification of the formulation of physical laws by using the natural systems of units is bought at the cost of replacing the physical equations by numerical one. In the used units, the numerical equations relates among themselves not the physical quantities but only their numerical values. A possibility of comparison and check of the dimension of the considered physical quantities is lost. Due to preservation of the same lettering a substitution of physical equations by the numerical one is masked and there is an illusion of operating with real physical quantities and equations. Arises abstractness of used equations and the theory as a whole."

mula (2) may serve as a vivid illustration.

In the vast literature devoted to pair production in electromagnetic fields, we have not ever met a graphical representation of (2). Before we draw the graph $\bar{w}^{(1/2)}$ as a function of β , one need to choose the system of units. In [1] Schwinger likely used the Heaviside system of units, the most common in scientific literature of the time. In (2) the dimension of the probability density is $\text{cm}^{-3}\cdot\text{s}^{-1}$ and the dimension of electric field V/cm . We need to restore the "invisible" conversion factor to match the dimensions on both sides. Let us reproduce schematically the logic of calculations led to (1) in order to recover all constants \hbar and c and bring the final result in the SI units⁴.

Suppose we observe during a period of time ΔT for a three dimensional space volume ΔV filled with a homogeneous and static electric field E . We write the wave function of the vacuum as $\Psi = \exp(iS/\hbar)$, where the action S is related to the density of the Lagrangian function \mathcal{L} via $S = \mathcal{L}\Delta V\Delta T$. As a result, the probability that the vacuum will be stable is $W_v = |\Psi|^2 = \exp(-2\text{Im}\mathcal{L}\Delta V\Delta T/\hbar)$, and the probability of pair creation in the field is

$$W_p = 1 - W_v = 1 - \exp\left(-\frac{2}{\hbar}\text{Im}\mathcal{L}\Delta V\Delta T\right). \quad (8)$$

In the weak-field approximation, keeping only the first term in the expansion of the exponential, we reproduce the Schwinger result,

$$W_p^{(s)} \approx \frac{2}{\hbar}\text{Im}\mathcal{L}\Delta T\Delta V, \quad (9)$$

that is, the probability of pair production per unit time per unit volume is $\bar{w}^{(s)} = 2\text{Im}\mathcal{L}/\hbar$. The dimensions of the functions S and \mathcal{L} are fixed by the above formulas allowing to restore the required number of the desired constants. Thus, $E_{cr} = m^2c^3/e\hbar$ and

$$\bar{w}^{(1/2)}(\beta) = w_0\beta^2e^{-\pi/\beta} \quad (10)$$

where

$$w_0 = \frac{e^2}{4\pi\hbar c} \frac{E_{cr}^2}{\pi^2\hbar} = \frac{m^4c^5}{4\pi^3\hbar^4}.$$

For e^+e^- pairs, after substituting numerical values of the constants one find that in (2) behind the seemingly small pre-exponential factor is "hidden" the very large dimensional constant $w_0 = 1.087036 \times 10^{50} \text{ cm}^{-3}\cdot\text{s}^{-1}$. Consequently, the earlier qualitative conclusion about the smallness of $\bar{w}^{(1/2)}$ may not be entirely correct⁵.

In Fig. 2 by open dots shown $W_p^{(1/2)}$ calculated with (9) and (10) with $\Delta T\Delta V = 1 \text{ cm}^3\cdot\text{s}$. Taking into account the magnitude of w_0 , the result is not so surprising, despite the fact that the field strength is much lower than the critical value. As follows from Fig. 2, the Schwinger formula is valid only for fields of $E < \beta_c E_{cr}$. At higher field, the probability of pair production exceeds unity.

From (10) we find β_c at which $W_p^{(1/2)} = 1$,

$$\exp\{\ln(\beta_c^2 w_0 \Delta T \Delta V) - \frac{\pi}{\beta_c}\} = 1. \quad (11)$$

Let us denote $A = \frac{\pi}{2}\sqrt{w_0\Delta T\Delta V}$ and sequentially making change of variables $\sqrt{w_0\Delta T\Delta V}\beta_c = z$ and $z = e^W$, we get an equation

$$W(A)e^{W(A)} = A \quad (12)$$

for the Lambert $W(A)$ function. The Lambert function has the same fate as that of the dilogarithm. Interest in it was revived in the last quarter of the 20th century [10], [11]. The required β_c is obtained from

$$\beta_c = \frac{1}{\sqrt{w_0\Delta T\Delta V}}e^{W(A)}.$$

For numerical estimates we use the asymptotic expansion of $W(A)$ [10]

$$W(A) = L_1 - L_2 + \frac{L_2}{L_1} + \frac{L_2(L_2 - 2)}{2L_1^2} + \frac{L_2(2L_2^2 - 9L_2 + 6)}{6L_1^3} + \mathcal{O}\left(\left\{\frac{L_2}{L_1}\right\}^4\right), \quad (13)$$

where $L_1 = \ln A$, $L_2 = \ln \ln A$. Thus, $\beta_c(W_p = 1) \approx 0.02905$ at $\Delta V\Delta T = 1 \text{ cm}^3\cdot\text{s}$.

The reason on which at $\beta > \beta_c$ the unitarity is violated, lies in the use of the weak field approximation leading to (9). To restore the unitarity, it is necessary to take a step back to (8), which is free of the problem. In (8), $2\text{Im}\mathcal{L}/\hbar$ should be replaced by $\bar{w}^{(s)}$ from (5) and take into account the new system of units (10). After completing the necessary substitutions, we obtain the corrected formula for the probability of pair creation in the volume ΔV

$$W_p^{(s)}(\beta) = 1 - \exp\left\{\pm\left(s + \frac{1}{2}\right)w_0\beta^2\text{Li}_2(\pm e^{-\pi/\beta})\Delta V\Delta T\right\} \quad (14)$$

during the observation period ΔT . In (14) already makes sense to substitute the exact form we found for the Schwinger series, because we are not limited by the weakness of the field⁶. It should be emphasized that (14) explicitly demonstrates how the probability of the process depends on the size of the space-time volume. But this

⁴ However, in the following, as the unit of length is more convenient to use 1 cm.

⁵ Our expression for w_0 coincides with the analogous scaling constant of Refs [8, 9].

⁶ Provided that the two-loop correction [12] to the Heisenberg-Euler Lagrangian density gives a small contribution.

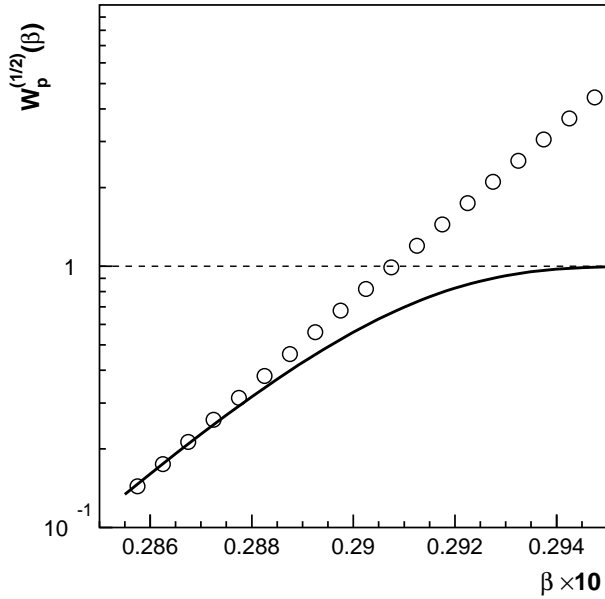


FIG. 2: The probability of an e^+e^- pair production in an unit 4-volume, $\Delta V \Delta T = 1 \text{ cm}^3 \cdot \text{s}$, as a function of the reduced field β . The open dots shows the result of calculation by equation (9), violating the unitarity. The solid curve shows the calculation by (14).

dependence is extremely weak if compared with the dependence on E . For example, if at a given field strength the probability of pair creation is 95%, and we wish to reduce the vacuum field by the factor ten, keeping the same particle production rate, then it is necessary to increase the field space-time volume in $10^2 \cdot \exp(\frac{2\pi}{\beta}) \approx 10^{420}$ times, that is, literally to the cosmological size.

In Fig. 2 by the solid curve shown the probability of an

e^+e^- pair production calculated by means of (14) with $\Delta V \Delta T = 1 \text{ cm}^3 \cdot \text{s}$. As is evident from the figure, $W_p^{(1/2)}$ increases rapidly within a very narrow range of variation of E . Indeed, when the field strength is 2.75% of E_{cr} the pair production probability is less than 1%, but in the field of strength 2.95% of E_{cr} , e^+e^- pairs are produced with almost 100% probability. This behavior is typical for tunneling processes.

4. In the present article we prove that the Schwinger series (1) for the probability of a pair of bosons and fermions production in a uniform and static electric field allows to be represented in terms of the dilogarithm. In a weak field, $\beta < 0.5$, this series with a very high accuracy is described by the first term. With increasing β the remainder of the series begins to play an important role, making the correction of 6% at $\beta = 2$. The restoration of all constants \hbar and c in the Schwinger formula reveals a very large pre-exponential factor $w_0 = 1.087036 \times 10^{50} \text{ cm}^{-3} \cdot \text{s}^{-1}$. Its presence determines a sharp increase of the pair production probability within a very narrow range of field intensities and in the weak field approximation leads to a violation of unitarity at $E > 0.0291 E_{cr}$. Equation (14) corrects this defect, satisfies the unitarity condition and includes the space-time volume of the process. Numerical estimates show that in the field of strength 2.95% of E_{cr} within the four-dimensional volume of $1 \text{ cm}^3 \cdot \text{sec}$, the probability of an e^+e^- pair production is close to 100%.

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- [1] J. Schwinger, Phys. Rev. **82**, 664 (1951).
 - [2] F. Sauter, Zs. Phys. **69**, 742 (1931).
 - [3] A. Casher, H. Neuberger and S. Nussinov, Phys. Rev. **D20**, 179 (1979).
 - [4] B. Andersson, G. Gustafson, G. Ingelman and T. Sjöstrand, Phys. Rep. **97**, 31 (1983).
 - [5] Bo Andersson, *The Lund Model* (Cambridge, Cambridge University Press, 1998).
 - [6] A. N. Kirillov, *Dilogarithm identities*, Prog. Theor. Phys. Suppl., **118**, 61 (1995); arXiv:hep-th/9408113, 1994.
 - [7] A. D. Vlasov and B. P. Murin, *Units of physical quantities in science and technology* (in Russian) (Moscow, Energoatomizdat, 1990).
 - [8] J. S. Heyl and L. Hernquist, Phys. Rev. **D55**, 2449 (1997).
 - [9] A. Ringwald, Phys. Lett. **B 510**, 107 (2001).
 - [10] R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey and D. E. Knuth, *On the Lambert W function*, Adv. Comp. Math. **5**, 329 (1996).
 - [11] A. E. Dubinov, I. D. Dubinova and S. K. Saikov, *The Lambert W-function and its application in mathematical physics problems* (in Russian) (Sarov, FGUP "RFYaC-VNIIEF", 2006).
 - [12] V. I. Ritus, Zh. Eksp. Teor. Fiz. **69**, 1517 (1975).